

Application of equivalence method to Monge-Ampère equations

Elliptic case

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Abstract

The application of equivalence method to classify Monge-Ampère system leads to three orbits, parabolic case, hyperbolic case and elliptic case wich correspond to three types of Monge-Ampère systems. In this paper we will study the elliptic case and give a presentation of the group as a complex group.

Keywords: Exterior Differential Systems, Equivalence problem, Monge-Ampère equations.

1 Introduction

Cartan's method to state the equivalence problem devolopped by Elie Cartan in the years 1905-1910 more recently in [9], is a crucial tool. We are here interested in its application to the study of Monge-Ampère equation in 2 variables. Hence following the work of R. Bryant, D. Grossman and P. Griffiths in the years 1997-1998 [2] in order to clarify the strategy of Cartan: Given a 5-dimensional contact manifold (\mathcal{M}, I) they applied this method to some Monge-Ampère equations given in [6] $\varepsilon = \{\theta, d\theta, \Psi\}$ for $\theta \in \Gamma(I)$ and Ψ is a 2-form. We may assume $d\theta \wedge \Psi = 0 \bmod \{I\}$. On the contact manifold \mathcal{M} , one can locally find a coframing $\eta = (\eta^a)$ such that $\eta^0 \in \Gamma(I)$ and

$$d\eta^0 = \eta^1 \wedge \eta^2 + \eta^3 + \eta^4 \bmod \{I\}, \quad (1)$$

then we can write $\Psi = \frac{1}{2}b_{ij}\eta^i \wedge \eta^j$. We can find that there are three types of Monge-Ampère systems:

1. If $\Psi \wedge \Psi$ is a negative multiple of $d\eta^0 \wedge d\eta^0$, then the local coframing η my be chosen so that in addition to (1),

$$\Psi = \eta^1 \wedge \eta^2 - \eta^3 + \eta^4 \bmod \{I\};$$

for a classical problem, this occurs when the Euler-Lagrange PDE is hyperbolic.

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2. If $\Psi \wedge \Psi = 0$, then the local coframing η may be chosen so that in addition to (1),

$$\Psi = \eta^1 \wedge \eta^3 \bmod \{I\};$$

for a classical problem, this occurs when the Euler-Lagrange PDE is parabolic.

3. If $\Psi \wedge \Psi$ is a positive multiple of $d\eta^0 \wedge d\eta^0$, then the local coframing η may be chosen so that in addition to (1),

$$\Psi = \eta^1 \wedge \eta^4 - \eta^3 + \eta^2 \bmod \{I\},$$

for a classical problem, this occurs when the Euler-Lagrange PDE is elliptic.

R. Bryant, D. Grossman and P. Griffiths applied the equivalence method [7] to study the hyperbolic case. Our aim here is to study the elliptic case in which we apply the equivalence method, using a presentation of the acting group as a complex group. We determined the case when an elliptic Monge-Ampère system is locally equivalent to the Monge-Ampère system for the linear homogeneous Laplace equations and the case when it is locally equivalent to an Euler-Lagrange system.

2 Monge-Ampère System

Denote by $\mathcal{J}^1(\mathbb{R}^2, \mathbb{R})$ be the first order jet space

$$\mathcal{J}^1(\mathbb{R}^2, \mathbb{R}) := \{(x^1, x^2, z, p_1, p_2) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2\}.$$

For all smooth function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, we associate the graph of u by $\Sigma := j^1u(\mathbb{R}^2) \subset \mathcal{M}$ with

$$j^1u : \mathbb{R}^2 \rightarrow \mathcal{J}^1(\mathbb{R}^2, \mathbb{R}).$$

We have $z \circ (\mathcal{J}^1(u)) = u$ and for $1 \leq a \leq 2$, $p_a \circ (\mathcal{J}^1(u)) = \frac{\partial u}{\partial x^a}$, then

$$\Sigma := \left\{ \left(x^1, x^2, u(x), \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2} \right), x \in \mathbb{R}^2 \right\},$$

is a smooth submanifold of $\mathcal{J}^1(\mathbb{R}^2, \mathbb{R})$. We define the one-form θ which is not closed by

$$\theta = dz - p_1 dx^1 - p_2 dx^2.$$

If Ψ is a two form over $\mathcal{J}^1(\mathbb{R}^2, \mathbb{R})$, then

$$\begin{aligned} \Psi = & \Psi_{p_1 p_2} dp_1 \wedge dp_2 + \Psi_{p_1 x^2} dp_1 \wedge dx^2 + \Psi_{p_2 x^2} dp_2 \wedge dx^2 + \Psi_{p_1 x^1} dp_1 \wedge dx^1 \\ & + \Psi_{p_2 x^1} dp_2 \wedge dx^1 + \Psi_{x^1 x^2} dx^1 \wedge dx^2 \quad \bmod(\theta). \end{aligned}$$

A Monge-Ampère equation reads

$$\begin{cases} \Psi|_{\Sigma} = 0, \\ \theta|_{\Sigma} = 0 \quad (\Rightarrow d\theta|_{\Sigma} = 0), \\ dx^1 \wedge dx^2|_{\Sigma} \neq 0, \end{cases}$$

Along of Σ , we have

$$\begin{aligned} & \Psi_{p_1 p_2} \left[\frac{\partial^2 u}{(\partial x^1)^2} \frac{\partial^2 u}{(\partial x^2)^2} - \left(\frac{\partial^2 u}{\partial x^1 \partial x^2} \right)^2 \right] + \Psi_{p_1 x^2} \frac{\partial^2 u}{(\partial x^1)^2} + \Psi_{p_2 x^1} \frac{\partial^2 u}{(\partial x^2)^2} \\ & + (\Psi_{p_1 x^1} + \Psi_{p_2 x^2}) \frac{\partial^2 u}{\partial x^1 \partial x^2} + \Psi_{x^1 x^2} \left(x^a, u(x), \frac{\partial u}{\partial x^a} \right) = 0. \end{aligned}$$

Without loss of generality we can normalize by assuming Ψ , we have

$$\Psi_{p_1 x^1} = \Psi_{p_2 x^2} \Leftrightarrow d\theta \wedge \Psi = 0 \pmod{(\theta)},$$

Hence the data of Mong-Ampère equation are :

$$\begin{cases} \theta \in \Omega^1(\mathcal{M}) \text{ and } \Psi \in \Omega^2(\mathcal{M}), \\ \Psi \wedge d\theta = 0 \pmod{(\theta)}, \\ \theta \wedge d\theta \wedge d\theta \neq 0, \end{cases}$$

Lemma 2.1. Let $\Lambda = L(x, z, p)dx$ be a 1-form, for $x \in \mathbb{R}^n$, $z = u(x)$ and $p = (p_i) = \frac{\partial u}{\partial x^i}$. Over $\mathcal{J}^1(\mathbb{R}^n, \mathbb{R})$; we introduce the contact form, $\theta = dz - p_i dx^i$. If $\Sigma \subset \mathcal{J}^1(\mathbb{R}^n, \mathbb{R})$ define by $\Sigma = j^1 u(\Omega) = \{(x, u(x), p); \ x \in \Omega \subset \mathbb{R}^n\}$ with,

$$\begin{cases} dx|_{\Sigma} \neq 0, \\ \theta|_{\Sigma} \neq 0, \end{cases}$$

Then there exists a unique form Ξ and a 1-form α as

$$d\Lambda = \theta \wedge \Xi + d\alpha,$$

Remark 2.2. With some additional conditions, theses variational problems become a Euler-Lagrange equation of type Monge-Ampère.

1. The lemma 2.1 is true for any form $\Lambda \in \Omega^n(\mathcal{M})$.
2. Along Σ , Euler-Lagrange equations of the action $\int_{\Sigma} \Lambda$, where $\theta|_{\Sigma} = 0$, are given by:

$$\frac{\partial L}{\partial z} dx - \frac{d}{dx^i} \left(\frac{\partial L}{\partial p_i} \right) = 0.$$

Definition 2.3. The unique form $\Pi := \theta \wedge \Psi$ is called *Poincaré-Cartan form*.

Theorem 2.4. *A Monge-Ampère system $(\mathcal{M}, \varepsilon = \{\theta, d\theta, \Psi\})$ is locally equivalent to an Euler-Lagrange system if and only if*

$$d\Psi := d(\theta \wedge \Psi) = \varphi \wedge \Psi,$$

with $d\varphi \equiv 0 \pmod{\{\theta, d\theta, \Psi\}}$.

Proof. See [2] page 18-19. □

Theorem 2.5. *(Darboux) Given a 1-form θ as $\theta \wedge d\theta \neq 0$. So it is written in the form*

$$\theta = dz - p_1 dx^1 - p_2 dx^2.$$

3 Equivalence problem

We apply the equivalence problem to classify the Monge-Ampère system $\varepsilon = (\mathcal{M}, \theta, \Psi)$ satisfying

$$\begin{cases} \theta \wedge d\theta \wedge d\theta \neq 0, \\ \Psi \wedge d\theta = 0 \pmod{(\theta)}, \end{cases}$$

by comparing it with another system $\tilde{\varepsilon} = (\tilde{\mathcal{M}}, \tilde{\theta}, \tilde{\Psi})$, by looking at a diffeomorphism $\varphi : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ such that

$$\begin{cases} \varphi^* \tilde{\theta} = \theta \\ \varphi^* \tilde{\Psi} = \Psi \end{cases}$$

3.1 Preliminaries

Let $\eta = \alpha\theta \neq 0$, $\alpha \neq 0$. Locally, by Darboux theorem, we can find 1-forms $\eta^0, \eta^1, \eta^2, \eta^3, \eta^4$ such that

$$d\eta^0 = \eta^1 \wedge \eta^2 + \eta^3 \wedge \eta^4 \pmod{(\theta)}, \quad (2)$$

There exist functions b_{ij} such that $\Psi = \frac{1}{2}b_{ij}\eta^i \wedge \eta^j$, and as $\Psi \wedge d\eta^0 = 0 \pmod{(\theta)}$, then

$$b_{12} + b_{34} = 0,$$

We will study the conditions imposed by $\eta = (\eta^1, \eta^2, \eta^3, \eta^4)$ such that (2) be checked. So there are three non-zero orbits with we call: negative space, null space and positive space.

1. If $\Psi \wedge \Psi$ is a negative multiple of $d\eta^0 \wedge d\eta^0$, then the local coframing η may be chosen so that in addition of (2),

$$\Psi = \eta^1 \wedge \eta^2 - \eta^3 \wedge \eta^4 \pmod{(\theta)},$$

for a classical variational problem, this occurs when the Euler-Lagrange PDE is hyperbolic.

2. If $\Psi \wedge \Psi = 0$, then η may be chosen so that

$$\Psi = \eta^1 \wedge \eta^3 \mod(\theta),$$

for a classical variational problem, this occurs when the Monge-Ampère PDE is parabolic.

3. If $\Psi \wedge \Psi$ is a positive multiple of $d\eta^0 \wedge d\eta^0$, then the local coframing η may be chosen so that in addition of (2),

$$\Psi = \eta^1 \wedge \eta^4 - \eta^3 \wedge \eta^2 \mod(\theta),$$

for a classical variational problem, this occurs when the Monge-Ampère PDE is elliptic.

In the following we will study the elliptic case, we look at the following conditions

$$\left\{ \begin{array}{l} \eta = \alpha\theta \neq 0 \\ d\eta^0 = \eta^1 \wedge \eta^2 + \eta^3 \wedge \eta^4 \mod(\theta), \\ \Psi = \eta^1 \wedge \eta^4 - \eta^3 \wedge \eta^2 \mod(\theta), \end{array} \right. \quad (3)$$

3.2 An algebra preliminary

For $\omega := (\omega^1, \omega^2, \omega^3, \omega^4) \in (\mathbb{R}^4)^*$, we consider the symmetric non-degenerate function

$$\langle \cdot, \cdot \rangle : \Lambda^2(\mathbb{R}^4)^* \times \Lambda^2(\mathbb{R}^4)^* \longrightarrow \mathbb{R},$$

$$(\alpha, \beta) \longmapsto \langle \alpha, \beta \rangle := \frac{\alpha \wedge \beta}{\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4},$$

The Lie algebra $SL(4, \mathbb{R})$ acts on $\Lambda^2(\mathbb{R}^4)^* \simeq \mathbb{R}^6$ through the action $\forall g \in SL(4, \mathbb{R})$

$$q : \Lambda^2(\mathbb{R}^4)^* \longrightarrow \Lambda^2(\mathbb{R}^4)^*,$$

$$\alpha \longmapsto q(\alpha) := g^* \alpha,$$

q_g is a quadratic form in g and we have

$$\langle q(\alpha), q(\beta) \rangle = \langle g^* \alpha, g^* \beta \rangle = \langle \alpha, \beta \rangle.$$

We want to represent $SL(4, \mathbb{R}) = \text{Sp}(\mathbb{R}^{3,3}) := \text{Sp}(3, 3)^1$ on \mathbb{R}^6 . Let $G := SO(\Lambda^2(\mathbb{R}^4)^*, \langle \cdot, \cdot \rangle) \subset GL(6, \mathbb{R})$.

Denote

$$\Phi : SL(4, \mathbb{R}) \longrightarrow G,$$

$$\alpha \longmapsto q(\alpha),$$

¹“Spin” is a notation used by physicists. ($\text{Spin}(1, 3) = SL(2, \mathbb{C})$).

A basis $(\alpha_L^1, \alpha_L^2, \alpha_L^3, \alpha_R^1, \alpha_R^2, \alpha_R^3)$ of $\Lambda^2(\mathbb{R}^4)^*$ given by

$$\begin{cases} \alpha_L^1 = \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4, \\ \alpha_L^2 = \omega^1 \wedge \omega^3 + \omega^4 \wedge \omega^2, \\ \alpha_L^3 = \omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3, \\ \alpha_R^1 = \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4, \\ \alpha_R^2 = \omega^1 \wedge \omega^3 - \omega^4 \wedge \omega^2, \\ \alpha_R^3 = \omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3, \end{cases}$$

We have $\forall a, b \in \{1, 2, 3\}$

$$\langle \alpha_L^a, \alpha_L^b \rangle = 2\delta_b^a, \quad \langle \alpha_R^a, \alpha_R^b \rangle = -2\delta_b^a \text{ and } \langle \alpha_L^a, \alpha_R^b \rangle = 0,$$

The signature of Φ is $(3, 3)$ and $SO(\Lambda^2(\mathbb{R}^4)^*, \langle \cdot, \cdot \rangle) \subset SL(4, \mathbb{R})$ and since $\dim SL(4, \mathbb{R}) = \dim SO(\Lambda^2(\mathbb{R}^4)^*, \langle \cdot, \cdot \rangle) = 15$, we have

$$G = \text{Spin}(3, 3),$$

The first step of the equivalence problem method is to find a group G_0 and the associated G_0 -structure B_0 . Here G_0 preserves (3) , we define

$$G_0 = G_{\text{ellip}} = \{g \in SL(4, \mathbb{R}), \quad g^* \alpha_L^1 = \alpha_L^1; \quad g^* \alpha_L^3 = \alpha_L^3\},$$

For all ξ in the Lie algebra $\mathfrak{g}_{\text{ellip}} := \mathfrak{g}$, we have $\xi = (\xi)_{1 \leq i, j \leq 4} \in M(4, \mathbb{R})$ $\text{tr} \xi = 0$, and moreover, we have

$$\begin{cases} \xi \in M(4, \mathbb{R}), \quad \text{tr} \xi = 0, \\ \xi^* \alpha_L^1 = \alpha_L^1; \quad \xi^* \alpha_L^3 = \alpha_L^3, \end{cases} \quad (4)$$

Hence

$$\begin{cases} \xi_1^1 + \xi_2^2 + \xi_3^3 + \xi_4^4 = 0, \\ \xi_a^1 \omega^a \wedge \omega^2 + \xi_a^2 \omega^1 \wedge \omega^a + \xi_a^3 \omega^a \wedge \omega^4 + \xi_a^4 \omega^3 \wedge \omega^a = \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4, \\ \xi_a^1 \omega^a \wedge \omega^4 + \xi_a^4 \omega^1 \wedge \omega^a + \xi_a^2 \omega^a \wedge \omega^3 + \xi_a^3 \omega^2 \wedge \omega^a = \omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3, \end{cases}$$

Then there exist $a, b, c, d, e, f \in \mathbb{R}$ and a basis $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$ of \mathfrak{g} such as

$$\xi = a\xi_1 + b\xi_2 + c\xi_3 + d\xi_4 + e\xi_5 + f\xi_6,$$

with

$$\xi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \xi_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \xi_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xi_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \xi_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \xi_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Denote the \mathbb{R} -linear application

$$\Phi : \mathfrak{g} \longrightarrow sl(2, \mathbb{C}),$$

$$X, Y \longmapsto \Phi([X, Y]) = [\Phi(X), \Phi(Y)].$$

A basis of $sl(2, \mathbb{C})$ is $(h_0, e_0, f_0, h_1, e_1, f_1)$ such that

$$[h_0, h_1] = [e_0, e_1] = [f_0, f_1] = 0, \quad [h_a, e_b] = -2i^{a+b}e_0,$$

$$[h_a, f_b] = -2i^{a+b}f_0 \text{ and } [e_a, f_b] = -i^{a+b}h_0,$$

We can find relations of the same type on the basis of \mathfrak{g} .

Structure constants of \mathfrak{g}	Structure constants of $sl(2, \mathbb{C})$
$[\xi_a, \xi_{3+a}] = 0 \quad \text{for } a = 1, 2, 3$	$[h_0, h_1] = [e_0, e_1] = [f_0, f_1] = 0$
$[\xi_1, \xi_2] = -2\xi_2,$	$[h_0, e_0] = -2e_0,$
$[\xi_1, \xi_3] = 2\xi_3,$	$[h_0, f_1] = 2f_1,$
$[\xi_1, \xi_5] = -2\xi_5,$	$[h_0, e_1] = -2e_1,$
$[\xi_1, \xi_6] = 2\xi_6,$	$[h_0, f_0] = 2f_0,$
$[\xi_4, \xi_2] = -2\xi_5,$	$[h_1, e_0] = -2e_1,$
$[\xi_4, \xi_6] = 2\xi_3,$	$[h_1, f_0] = 2f_1,$
$[\xi_4, \xi_5] = 2\xi_2,$	$[h_1, e_1] = 2e_0,$
$[\xi_4, \xi_3] = -2\xi_6,$	$[h_1, f_1] = -2f_0,$
$[\xi_2, \xi_6] = -\xi_1,$	$[e_0, f_0] = -h_0,$
$[\xi_2, \xi_3] = -\xi_4,$	$[e_0, f_1] = -h_1,$
$[\xi_5, \xi_6] = -\xi_4,$	$[e_1, f_0] = -h_1,$
$[\xi_5, \xi_3] = \xi_1,$	$[e_1, f_1] = h_0,$

Table 1: Comparison of the structure constants.

We have this correspondence

$$\xi_1 \longleftrightarrow h_0,$$

$$\xi_2 \longleftrightarrow e_0,$$

$$\xi_3 \longleftrightarrow f_1,$$

$$\xi_4 \longleftrightarrow h_1,$$

$$\xi_5 \longleftrightarrow e_1,$$

$$\xi_6 \longleftrightarrow f_0,$$

Denote the linear map

$$T : \mathbb{R}^4 \longrightarrow \mathbb{C}^2$$

$$X = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \longmapsto \begin{pmatrix} x^3 + ix^1 \\ x^2 + ix^4 \end{pmatrix},$$

We can show that $\forall 1 \leq i \leq 6$

$$T(\xi_i X) = \Phi(\xi_i)T(X),$$

3.3 Back to the equivalence problem

For $\omega = \begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix}$ and $\pi = \begin{pmatrix} \pi^0 \\ \pi^1 \\ \pi^2 \\ \bar{\pi}^1 \\ \bar{\pi}^2 \end{pmatrix}$ two vector valued 1-forms, $P \in M(5, \mathbb{C})$ we set $\omega = P\pi$, where

$$\begin{cases} \pi^0 = \omega^0, \\ \pi^1 = \omega^3 + i\omega^1, \\ \pi^2 = \omega^2 + i\omega^4, \\ \bar{\pi}^1 = \omega^3 - i\omega^1, \\ \bar{\pi}^2 = \omega^2 - i\omega^4, \end{cases}$$

Consider $\forall M_{\mathfrak{h}} = (a_j^i)_{1 \leq i, j \leq 4} \in \mathfrak{g}$, we note $M \in M(5, \mathbb{C})$ by

$$M = \begin{pmatrix} a_0^0 & 0 \\ 0 & M_{\mathfrak{h}} \end{pmatrix},$$

Introducing the torsion τ given by $\tau = d\omega + \varphi \wedge \omega$, we obtain

$$P^{-1}\tau = d\pi + \psi \wedge \pi, \tag{5}$$

where $\psi = P^{-1}\varphi P$. Taking that $M_{\mathfrak{t}} = (a_j^i)_{1 \leq i, j \leq 4} \in \mathfrak{g}$, thus

$$P^{-1}MP = \begin{pmatrix} a_0^0 & 0 & 0 & 0 & 0 \\ 0 & a_1^1 + ia_3^1 & a_4^1 + ia_2^1 & 0 & 0 \\ 0 & a_3^2 - ia_1^2 & a_2^2 - ia_4^2 & 0 & 0 \\ 0 & 0 & 0 & a_3^3 + ia_1^3 & a_2^3 + ia_4^3 \\ 0 & 0 & 0 & a_1^4 - ia_3^4 & a_4^4 - ia_2^4 \end{pmatrix},$$

Proposition 3.1. *Let $(\mathcal{M}^5, \varepsilon)$ an elliptic Monge-Ampère system. The adapted coframings are the sections of G_0 -structure on \mathcal{M} , where G_0 is the smallest subgroup generated by all matrices of size $(1, 2, 2)$ of the form*

$$\begin{pmatrix} a_0^0 & 0 & 0 \\ C & A & 0 \\ \bar{C} & 0 & \bar{A} \end{pmatrix}, \quad (6)$$

where $A \in sl(2, \mathbb{C})$ and $\det A = a_0^0 \neq 0$.

Proof. The sections of G -structure adapted (3) are of the form (6). \square

According to this proposition we can pass to the second step “Calculation of the structure equations”, consider

$$\psi = \begin{pmatrix} \psi_0^0 & 0 & 0 & 0 & 0 \\ \psi_0^1 & \psi_1^1 & \psi_2^1 & 0 & 0 \\ \psi_0^2 & \psi_1^2 & \psi_2^2 & 0 & 0 \\ \bar{\psi}_0^1 & 0 & 0 & \bar{\psi}_1^1 & \bar{\psi}_2^1 \\ \bar{\psi}_0^2 & 0 & 0 & \bar{\psi}_1^2 & \bar{\psi}_2^2 \end{pmatrix}, \quad (7)$$

where $\psi_1^1 + \psi_2^2 = \bar{\psi}_1^1 + \bar{\psi}_2^2 = \psi_0^0$.

We assume that $P^{-1}\tau = \begin{pmatrix} \tau^0 \\ \tau^1 \\ \tau^2 \\ \bar{\tau}^1 \\ \bar{\tau}^2 \end{pmatrix}$, where

$$\tau^0 := d\pi^0 + \psi_0^0 \wedge \pi^0 = \frac{i}{2}(\bar{\pi}^1 \wedge \bar{\pi}^2 - \pi^1 \wedge \pi^2),$$

and for $i = 1, 2, 3, 4$,

$$\begin{aligned} \tau^i &= T_{12}^i \pi^1 \wedge \pi^2 + T_{1\bar{1}}^i \pi^1 \wedge \bar{\pi}^1 + T_{1\bar{2}}^i \pi^1 \wedge \bar{\pi}^2 + T_{2\bar{1}}^i \pi^2 \wedge \bar{\pi}^1 + T_{2\bar{2}}^i \pi^2 \wedge \bar{\pi}^2 + T_{1\bar{2}}^i \bar{\pi}^1 \wedge \bar{\pi}^2 \\ &\quad + T_{01}^i \pi^0 \wedge \pi^1 + T_{02}^i \pi^0 \wedge \pi^2 + T_{0\bar{1}}^i \pi^0 \wedge \bar{\pi}^1 + T_{0\bar{2}}^i \pi^0 \wedge \bar{\pi}^2. \end{aligned}$$

This produces the structure equations

$$\begin{cases} d\pi^0 = -\psi_0^0 \wedge \pi^0 + \frac{i}{2}(\bar{\pi}^1 \wedge \bar{\pi}^2 - \pi^1 \wedge \pi^2), \\ d\pi^1 = -\psi_0^1 \wedge \pi^0 - \psi_1^1 \wedge \pi^1 - \psi_2^1 \wedge \pi^2 + \tau^1, \\ d\pi^2 = -\psi_0^2 \wedge \pi^0 - \psi_1^2 \wedge \pi^1 - \psi_2^2 \wedge \pi^2 + \tau^2, \\ d\bar{\pi}^1 = -\bar{\psi}_0^1 \wedge \pi^0 - \bar{\psi}_1^1 \wedge \bar{\pi}^1 - \bar{\psi}_2^1 \wedge \bar{\pi}^2 + \bar{\tau}^1, \\ d\bar{\pi}^2 = -\bar{\psi}_0^2 \wedge \pi^0 - \bar{\psi}_1^2 \wedge \bar{\pi}^1 - \bar{\psi}_2^2 \wedge \bar{\pi}^2 + \bar{\tau}^2, \end{cases} \quad (8)$$

Now we go to the second step which allows us to absorb the maximum of torsion in (8) respecting $\psi_1^1 + \psi_2^2 = \bar{\psi}_1^1 + \bar{\psi}_2^2 = \psi_0^0$. First, by change the form $\psi_0^i \leftarrow \psi_0^i - T_{0*}^i \pi^*$ we can consider²

$$T_{0*}^i = 0.$$

By a change of ψ_2^1 and ψ_1^2 , we can write

$$T_{21}^1 = T_{22}^1 = T_{12}^1 = T_{11}^2 = T_{12}^2 = T_{22}^2 = 0,$$

Respecting $\psi_1^1 + \psi_2^2 = \bar{\psi}_1^1 + \bar{\psi}_2^2 = \psi_0^0$, we can write

$$T_{11}^1 = T_{21}^2 = V_1 \text{ and } T_{22}^2 = T_{12}^1 = V_2,$$

Thus (8) becomes

$$\begin{cases} d\pi^0 = -\psi_0^0 \wedge \pi^0 + \frac{i}{2}(\bar{\pi}^1 \wedge \bar{\pi}^2 - \pi^1 \wedge \pi^2), \\ d\pi^1 = -\psi_0^1 \wedge \pi^0 - \psi_1^1 \wedge \pi^1 - \psi_2^1 \wedge \pi^2 + V_1 \pi^1 \wedge \bar{\pi}^1 + V_2 \pi^1 \wedge \bar{\pi}^2 + U_1 \bar{\pi}^1 \wedge \bar{\pi}^2, \\ d\pi^2 = -\psi_0^2 \wedge \pi^0 - \psi_1^2 \wedge \pi^1 - \psi_2^2 \wedge \pi^2 + V_1 \pi^2 \wedge \bar{\pi}^1 + V_2 \pi^2 \wedge \bar{\pi}^2 + U_2 \bar{\pi}^1 \wedge \bar{\pi}^2, \\ d\bar{\pi}^1 = -\bar{\psi}_0^1 \wedge \pi^0 - \bar{\psi}_1^1 \wedge \bar{\pi}^1 - \bar{\psi}_2^1 \wedge \bar{\pi}^2 + \bar{V}_1 \bar{\pi}^1 \wedge \pi^1 + \bar{V}_2 \bar{\pi}^1 \wedge \pi^2 + \bar{U}_1 \pi^1 \wedge \pi^2, \\ d\bar{\pi}^2 = -\bar{\psi}_0^2 \wedge \pi^0 - \bar{\psi}_1^2 \wedge \bar{\pi}^1 - \bar{\psi}_2^2 \wedge \bar{\pi}^2 + \bar{V}_1 \bar{\pi}^2 \wedge \pi^1 + \bar{V}_2 \bar{\pi}^2 \wedge \pi^2 + \bar{U}_2 \pi^1 \wedge \pi^2, \end{cases} \quad (9)$$

Here V_i and U_i are the new coefficients of torsion which are expressed in terms of T_{**}^i .

After calculating $0 \equiv d(d\pi^0)$, we have

$$U_1 = -2\bar{V}_2, \quad U_2 = 2\bar{V}_1.$$

We calculate $d(d\pi^1) \equiv 0$ and $d(d\pi^2) \equiv 0$, thus we have the relation mod $\{\pi^0, \pi^1, \pi^2\}$.

$$0 \equiv d \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} \psi_0^1 \\ \psi_0^2 \end{pmatrix} + \begin{pmatrix} \psi_1^1 & \psi_2^1 \\ \psi_1^2 & \psi_2^2 \end{pmatrix} \cdot \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} - \psi_0^0 \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \quad (10)$$

Let G_1 -structure $B_1 \subset B_0$ in which $\tau^1 = \tau^2 = 0$ and φ_0^1, φ_0^2 are semi-basic, we consider the projector $\Phi : B_0 \rightarrow B_1$ such that, for $x \in B_0$ we associate $x.g_0$ is a submersion which respects fibers.

²*, * $\in \{1, 2, \bar{1}, \bar{2}\}$.

Thus G_1 is a sub-group acting over B_1 generated by matrices of the form

$$g_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \bar{A} \end{pmatrix}, \quad (11)$$

Denote by

$$\psi_0^i = P_0^i \pi^0 + P_*^i \pi^*, \text{ and } \bar{\psi}_0^i = \bar{P}_0^i \pi^0 + \bar{P}_*^i \pi^*$$

So the structure equations read

$$\begin{cases} d\pi^0 = -\psi_0^0 \wedge \pi^0 + \frac{i}{2}(\bar{\pi}^1 \wedge \bar{\pi}^2 - \pi^1 \wedge \pi^2), \\ d\pi^1 = -\psi_1^1 \wedge \pi^1 - \psi_2^1 \wedge \pi^2 - P_*^1 \pi^* \wedge \pi^0, \\ d\pi^2 = -\psi_1^2 \wedge \pi^1 - \psi_2^2 \wedge \pi^2 + P_*^2 \pi^* \wedge \pi^0, \end{cases} \quad (12)$$

We absorb the torsion, respecting the condition $\psi_1^1 + \psi_2^2 = \psi_0^0$

$$\begin{cases} d\pi^0 = -\psi_0^0 \wedge \pi^0 + \frac{i}{2}(\bar{\pi}^1 \wedge \bar{\pi}^2 - \pi^1 \wedge \pi^2), \\ d\pi^1 = -\psi_1^1 \wedge \pi^1 - \psi_2^1 \wedge \pi^2 - P\pi^1 \wedge \pi^0 - P_1^1 \bar{\pi}^1 \wedge \pi^0 - P_2^1 \bar{\pi}^2 \wedge \pi^0, \\ d\pi^2 = -\psi_1^2 \wedge \pi^1 - \psi_2^2 \wedge \pi^2 - P\pi^2 \wedge \pi^0 - P_1^2 \bar{\pi}^1 \wedge \pi^0 - P_2^2 \bar{\pi}^2 \wedge \pi^0, \end{cases} \quad (13)$$

Respecting $\psi_1^1 + \psi_2^2 = \bar{\psi}_1^1 + \bar{\psi}_2^2 = \psi_0^0$, we can show

$$P + \bar{P} = 0. \quad (14)$$

We have

$$\begin{aligned} 0 &= -d\psi_0^0 \wedge \pi^0 + \frac{i}{2}\psi_0^0 \wedge \bar{\pi}^1 \wedge \bar{\pi}^2 - \frac{i}{2}\psi_0^0 \wedge \pi^1 \wedge \pi^2 \\ &+ \frac{i}{2}d\bar{\pi}^1 \wedge \bar{\pi}^2 - \frac{i}{2}\bar{\pi}^1 \wedge d\bar{\pi}^2 - \frac{i}{2}d\pi^1 \wedge \pi^2 + \frac{i}{2}\pi^1 \wedge d\pi^2, \end{aligned}$$

thus

$$\begin{aligned} 2id\psi_0^0 \wedge \pi^0 &= (2\bar{P}\bar{\pi}^1 \wedge \bar{\pi}^2 - 2P\pi^1 \wedge \pi^2 - (P_1^2 + \bar{P}_1^2)\pi^1 \wedge \bar{\pi}^1 + (\bar{P}_1^1 - P_2^2)\pi^1 \wedge \bar{\pi}^2 \\ &+ (P_1^1 - \bar{P}_2^2)\pi^2 \wedge \bar{\pi}^1 + (\bar{P}_2^1 - P_2^1)\pi^2 \wedge \bar{\pi}^2) \wedge \pi^0, \end{aligned}$$

thus

$$P - \bar{P} = 0,$$

for (14), then we have

$$P = 0,$$

then (13) reads

$$\begin{cases} d\pi^0 = -\psi_0^0 \wedge \pi^0 + \frac{i}{2}(\bar{\pi}^1 \wedge \bar{\pi}^2 - \pi^1 \wedge \pi^2), \\ d\pi^1 = -\psi_1^1 \wedge \pi^1 - \psi_2^1 \wedge \pi^2 - P_1^1 \bar{\pi}^1 \wedge \pi^0 - P_2^1 \bar{\pi}^2 \wedge \pi^0, \\ d\pi^2 = -\psi_1^2 \wedge \pi^1 - \psi_2^2 \wedge \pi^2 - P_1^2 \bar{\pi}^1 \wedge \pi^0 - P_2^2 \bar{\pi}^2 \wedge \pi^0, \end{cases} \quad (15)$$

in particular

$$\begin{aligned} 2id\psi_0^0 &= -(P_1^2 + \bar{P}_1^2)\pi^1 \wedge \bar{\pi}^1 + (\bar{P}_1^1 - P_2^2)\pi^1 \wedge \bar{\pi}^2 \\ &\quad + (P_1^1 - \bar{P}_2^2)\pi^2 \wedge \bar{\pi}^1 + (\bar{P}_2^1 - P_2^2)\pi^2 \wedge \bar{\pi}^2. \end{aligned} \quad (16)$$

We define a pair of 2×2 matrix-valued functions on B_1 by

$$S_1 = \begin{pmatrix} P_1^1 + \bar{P}_2^2 & \bar{P}_2^1 + P_2^1 \\ P_1^2 - \bar{P}_1^2 & \bar{P}_1^1 + P_2^2 \end{pmatrix}, \quad S_2 = \begin{pmatrix} P_1^1 - \bar{P}_2^2 & \bar{P}_2^1 - P_2^1 \\ P_1^2 + \bar{P}_1^2 & \bar{P}_1^1 - P_2^2 \end{pmatrix}.$$

Theorem 3.2. *An elliptic Monge-Ampère system $(\mathcal{M}, \varepsilon)$ satisfies $S_1 = S_2 = 0$ if and only if it is locally equivalent to the Monge-Ampère system for the linear homogeneous Laplace equations.*

Proof. If $S_2 = 0$, then ψ_0^0 is closed, from (16), $S_2 = 0$ if and only if for some 1-form α we have

$$d\psi_0^0 = \alpha \wedge \pi^0.$$

But $dd = 0$, hence $0 \equiv -\alpha \wedge d\pi^0$, which gives us

$$\alpha \equiv 0 \pmod{\pi^0}$$

Conversely, if $d\psi_0^0 = 0$, then $S_2 = 0$. In case $S_1 = S_2 = 0$, then $d\psi_0^0 = 0$, thus we can locally find a function $\lambda > 0$ such that

$$\psi_0^0 = \lambda^{-1} d\lambda$$

In case $S_1 = S_2 = 0$ we can find

$$d(\pi^1 \wedge \pi^2) = -\psi_0^0 \pi^1 \wedge \pi^2$$

hence, we can write

$$d(\lambda\omega^1 \wedge \omega^4) = d(\lambda\omega^3 \wedge \omega^2) = d(\lambda\omega^1 \wedge \omega^2) = d(\lambda\omega^3 \wedge \omega^4) = 0$$

Then locally by (2.5) there exist functions x, y, p and q such that

$$-dp \wedge dx = \lambda\omega^1 \wedge \omega^2$$

$$-dq \wedge dy = \lambda\omega^3 \wedge \omega^4$$

$$-dp \wedge dy = \lambda\omega^1 \wedge \omega^4$$

$$-dq \wedge dx = \lambda\omega^3 \wedge \omega^2$$

Not that

$$d(\lambda\pi^0) = d(\lambda\omega^0) = \frac{i}{2}(\bar{\pi}^1 \wedge \bar{\pi}^2 - \pi^1 \wedge \pi^2) = \lambda(\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4) = -dp \wedge dx - dq \wedge dy$$

By Poincaré lemma, locally there is exist a function z , such that

$$\lambda\omega^0 = dz - pdx - qdy$$

Then, in local coordinates, our elliptic Monge-Ampère system is

$$\begin{aligned}\varepsilon &= \{\omega^0, \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4, \omega^1 \wedge \omega^4 - \omega^3 \wedge \omega^2\} \\ &= \{dz - pdx - qdy, -dp \wedge dx - dq \wedge dy, -dp \wedge dy + dq \wedge dx\}.\end{aligned}$$

□

It's natural to ask about the situation in wich $S_2 = 0$, but possibly $S_1 \neq 0$.

Theorem 3.3. *An elliptic Monge-Ampère system $(\mathcal{M}, \varepsilon)$ satisfies $S_2 = 0$ if and only if it is locally equivalent to an Euler-Lagrange system.*

Proof. The condition for ε to contain a Poincaré-Cartan form

$$\begin{aligned}\Pi &= \frac{1}{2}\lambda\pi^0 \wedge (\bar{\pi}^1 \wedge \bar{\pi}^2 + \pi^1 \wedge \pi^2). \\ &= \lambda\omega^0 \wedge (\omega^1 \wedge \omega^4 - \omega^3 \wedge \omega^2).\end{aligned}$$

We can assume that Π to be closed for some $\lambda > 0$ on B_1 . By differentiating then

$$0 = (d\lambda - 2\lambda\psi_0^0) \wedge \omega^0 \wedge (\omega^1 \wedge \omega^4 - \omega^3 \wedge \omega^2).$$

Exterior algebra, for some function μ , say

$$d\lambda - 2\lambda\psi_0^0 = \mu\lambda\omega^0.$$

In other words,

$$d(\log \lambda) - 2\psi_0^0 = \mu\omega^0.$$

Hence

$$d\psi_0^0 \equiv 0 \text{ mod } \{\omega^0\}.$$

But we know that

$$d\omega^0 = \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4.$$

(16), gives us $S_2 = 0$.

□

3.4 Remark in Cartan's test

Definition 3.4. If (π^0, π^1, π^2) be a lifted coframe, then the associated Exterior Differential System, with equivalence condition $\pi^0 \wedge \pi^1 \wedge \pi^2 \wedge \bar{\pi}^1 \wedge \bar{\pi}^2 \neq 0$, it is involutive if and only if it satisfies the Cartan's test.

To apply equivalence method to some problem there are several steps, one important is Cartan's test. If the problem is involutive we can conclude, if this is not the case, it is necessary to extend the system to continue. We begin, for example, to test the involution in the elliptic case. To find this there is a process to follow [7], in (15), we have $r = 5$ and $n = 3$; to find the reduced characters of Cartan, replace in (15) ψ_j^i by $z_{j0}^i \pi^0 + z_{j1}^i \pi^1 + z_{j2}^i \pi^2$, we can show

$$\begin{cases} z_{0j}^0 = 0 & j = 0, \dots, 2, \\ z_{10}^1 = 0, & z_{11}^1 = z_{12}^2, \\ z_{20}^1 = 0, & z_{12}^1 = z_{21}^1, z_{22}^1, \\ z_{10}^2 = 0, & z_{11}^2, \end{cases} \quad (17)$$

The four parameters $z_{11}^1, z_{12}^1, z_{22}^1, z_{11}^2$ can be chosen arbitrarily, thus the degree of indeterminacy $r^{(1)}$ of a lifted coframe is the number of free variables in the solution to the associated linear absorption system

$$r^{(1)} = 4.$$

Let be $X = (x^0, \dots, x^2) \in \mathbb{R}^3$ and the matrix M of size 3×4 define by

$$M(X) := M_k^l(X) := \sum_{j=0}^2 A_{jk}^l x^j, \quad l = 0, \dots, 2, \quad \binom{l}{k} \in \binom{0}{0,1}, \binom{1}{2}, \binom{2}{1},$$

where A_{jk}^l are coefficients define in (15). In other words

$$M(X) = \left(A_{0k}^l(x^0) + A_{1k}^l(x^1) + A_{2k}^l(x^2) \right)_{\substack{0 \leq l \leq 2 \\ \binom{l}{k} \in \binom{0}{0,1}, \binom{1}{2}, \binom{2}{1}}},$$

Thus

$$M(X) = \begin{pmatrix} -x^0 & 0 & 0 & 0 \\ 0 & -x^1 & -x^2 & 0 \\ -x^2 & x^2 & 0 & -x^1 \end{pmatrix}.$$

For $X = (-1, -1, 0)$, thus

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we denote by s'_1, \dots, s'_3 the reduced characters Cartan then

$$s'_1 = 3.$$

Now, for $X = (x^0, \dots, x^2)$ and $Y = (y^0, \dots, y^2)$, we have

$$\begin{pmatrix} M(X) \\ M(Y) \end{pmatrix} = \begin{pmatrix} -x^0 & 0 & 0 & 0 \\ 0 & -x^1 & -x^2 & 0 \\ -x^2 & x^2 & 0 & -x^1 \\ -y^0 & 0 & 0 & 0 \\ 0 & -y^1 & -y^2 & 0 \\ -y^2 & y^2 & 0 & -y^1 \end{pmatrix}.$$

For $X = (-1, -1, 0)$ and $Y = (0, 0, -1)$, we have $s'_1 + s'_2 = 4$, thus

$$s'_2 = 1.$$

Or we have $s'_1 + s'_2 + s'_3 = r = 4$, then

$$s'_3 = 0,$$

thus we have

$$s'_1 + 2s'_2 + 3s'_3 = 5 > r^{(1)} = 4.$$

Hence the system (15) is not satisfies Cartan's test, thus it's necessary to extend the system to continue. Note that before the step of normalizing, the system (3.1) satisfied Cartan's test, I gave a proof of this in my thesis [5]. This leads to further investigations.

Acknowledgement I am grateful to Frédéric Hélein who suggested to me this problem as a part of my Ph.D.

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